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The geometric phase in photon systems

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Abstract. By using the Maxwell equations in a Schrödinger-like form, the geometric phase in photon systems is calculated. This approach provides a unified way to discuss geometric phases in both photon (massless) and other massive particle systems. The concept of parameter space is not introduced in our discussion. So all complications caused by it can be avoided. In principle, this approach can used to calculate the geometric phase in any cyclic evolution of photon systems.

PACS. 03.65.Bz Foundations, theory of measurement, miscellaneous theories (including Aharonov-Bohm effect, Bell inequalities, Berry's phase) – 41.20.Bt Maxwell equations, time-varying fields, conservation laws – 42.90.+m Other topics in optics

1 Introduction

Berry pointed out [1] that a nondegenerate quantum state $|\psi(t)\rangle$ of a Hamiltonian $H(t)$ which varies adiabatically through a circuit C in parameter space acquires an additional phase $\gamma(C)$ which is only relative to the geometry of the parameter space, besides the "normal" dynamical phase

$$\phi_d = -\frac{1}{\hbar}\int \langle\psi(t)|H(t)|\psi(t)\rangle dt. \qquad (1)$$

$\gamma(C)$ is called the geometric phase, topological phase or Berry's phase. Immediately, Simon [2] gave a mathematical interpretation to the geometric phase as a $U(1)$ holonomy on a complex Hilbert line bundle. Since then the geometric phase (or Berry's phase) has been widely discussed and observed in various fields.

In Berry's original discussion, there were three constraints on the quantum system: (i) nondegenerate state; (ii) adiabatic and (iii) cyclic evolution. All the three constraints were removed later. The first one was removed by Wilczek and Zee [3]. They related the evolution and phases of a degenerate manifold to a non-Abelian Gauge, *i.e.*, $\gamma(C)$ is a $U(N)$ holonomy for an N-fold degenerate level.

The substantial generalization was developed by Aharonov and Anandan [4]. They argued what should be dealt with more fundamentally are circuits of the quantum system itself, rather than circuits of the Hamiltonian in a parameter space. This means that the concept of the parameter space is no longer necessary. In their formulation, a geometric phase factor β is introduced for any cyclic evolution of a quantum system as follows:

$$\beta = \int_0^\tau \langle\tilde{\psi}|i(d|\tilde{\psi}\rangle/dt)dt, \qquad (2)$$

where $|\tilde{\psi}(t)\rangle = e^{-if(t)}|\psi(t)\rangle$, $f(\tau)-f(0)=\phi$ and $|\psi(t)\rangle = e^{i\phi}|\psi(0)\rangle$. β is so-called the Aharonov-Anandan phase and the Berry's phase $\gamma(C)$ can be considered as its adiabatic limit.

The last one was removed by Samuel and Bhandari [5]. They pointed out that one can close a non-cyclic evolution with a geodesic and obtain the corresponding geometric phase.

On the experimental side, a series of striking experiments have also been carried out and their results provide direct or indirect evidences for the existence of geometric phases [6]. Among them the most remarkable one is to observe the angle of rotation of linearly polarized light propagating down a helically wound, single-mode optical fiber. This was suggested by Chiao and Wu [7] first and was carried through by Tomita and Chiao [8] immediately.

Photons are neutral, massless relativistic particles. They are quite different from other charged, massive ones. Photons cannot be described by the non-relativistic Schrödinger equation. It is well-known, the role of the Schrödinger equation in the calculation of the geometric phase is to provide a connection. So how do we calculate the geometric phase in photon systems without a connection? In order to solve this problem, one usually adopts two methods. One is to use so-called Pancharatnam's connection [9]. The other is to do so by analogy between the helicity and the spin magnetic moment [7]. In both cases, however, the calculation is not started from the Schrödinger equation. This means there is no unified

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way to discuss geometric phases in both systems of photons (or, more general, massless particles) and other massive ones.

In this paper, in terms of the Maxwell equations in a Schrödinger-like form [10], the geometric phase (in Aharonov-Anandan formulation) in photon systems is calculated. This approach provides a unified way to deal with both massless and massive particle systems. The concept of parameter space is not introduced in our discussion, so all complications and ambiguits caused by it, for example birefringence, elastooptical effects caused by the optical fibre, can be avoided.

2 The Maxwell equations in Schrödinger form

The spin 1, massless photon is described by the Maxwell equations. In order to discuss the geometric phase of the photon, we need to rewrite the Maxwell equations in a form similar to the Schrödinger equation. This can be realized by combining the electric field **E** and the magnetic field **B** into a complex column vector Ψ:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \tag{3}$$

where $\psi_i \equiv E_i + iB_i, (i = 1, 2, 3)$ and E_i and B_i are the components of **E** and **B** respectively.

The Maxwell equations in vacuum (without exterior source) are

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{4.1}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 0, \tag{4.2}$$

$$\nabla \cdot \mathbf{E} = 0, \tag{4.3}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{4.4}$$

Where we have set $c = 1$. Use equation (3), then equations (4.1, 4.2) can be combined to write as

$$\varepsilon_{ijk} \frac{\partial \psi_k}{\partial x_j} - i \frac{\partial \psi_i}{\partial t} = 0, \tag{5}$$

and equation (4.3, 4.4) as

$$\frac{\partial \psi_i}{\partial x_i} = 0, \tag{6}$$

where ε_{ijk} is the totally antisymmetric tensor in three dimensions.

Equation (6) can be considered as the initial condition of equation (5). This can be shown as below. Partial differentiating both side of equation (5) with x_i, we have

$$\varepsilon_{ijk} \frac{\partial^2 \psi_k}{\partial x_i \partial x_j} - i \frac{\partial}{\partial x_i} \left(\frac{\partial \psi_i}{\partial t} \right) = 0.$$

Due to the antisymmetry of ε_{ijk} and the symmetry of $\partial^2/\partial x_i \partial x_j$, the first term vanishes, *i.e.*

$$\frac{\partial}{\partial x_i} \frac{\partial \psi_i}{\partial t} = 0$$

or

$$\frac{\partial \psi_i}{\partial x_i} = const.$$

Letting $\partial \psi_i / \partial x_i = 0$ at $t = 0$, then equation (6) is just the initial condition of equation (5). This means that we need only to consider equation (5).

If set

$$(S^i)_{jk} \equiv -i\varepsilon_{ijk}; \quad p_i \equiv -i \frac{\partial}{\partial x_i}, \tag{7}$$

then equation (5) becomes

$$i \frac{\partial \psi}{\partial t} = (S^j)_{ik} p_j \psi_k. \tag{8}$$

It is easy to show

$$[S^i, S^j] = i\varepsilon_{ijk} S^k.$$

So

$$\mathbf{S} = S^1 \mathbf{e}_1 + S^2 \mathbf{e}_2 + S^3 \mathbf{e}_3$$

can be viewed as a spin operator. On the other hand,

$$\mathbf{p} = p_1 \mathbf{e}_1 + p_2 \mathbf{e}_2 + p_3 \mathbf{e}_3$$

looks like a momentum operator. So

$$H_{ik} \equiv (S^j)_{ik} p_j = -\varepsilon_{ijk} \partial_j \tag{9}$$

can be viewed as matrix elements of the Hamiltonian. Using equations (3, 9), equation (8) becomes

$$i \frac{\partial \psi_k}{\partial t} = H_{ik} \psi_k \tag{10}$$

or

$$i \frac{\partial \Psi}{\partial t} = H \Psi, \tag{11}$$

where

$$H = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}. \tag{12}$$

Equation (11) looks like a Schrödinger equation in the natural unit system ($c = \hbar = 1$), however, it is still the classical Maxwell equations (in complex form). H is not a true Hamiltonian either, but an operator defined by equations (7, 9). We should note that the "Hamiltonian" H, (Eq. (12)), does not possess the usual form of Hamiltonian: $H = \mathbf{p}^2/2m + V$. This fact indicates that photons are relativistic particles.

3 The calculation of the AA phase

The photon's AA phase β can be calculated in a straight forward way from equations (2, 11).

Photons are massless spin-1 particles, so their helicity only possesses two directions: parallel or anti-parallel to their momentum direction. From equation (9), H can be viewed as a helicity operator (*i.e.* $\mathbf{S}, \mathbf{P}$ are viewed as a spin and momentum operator respectively). The eigenvalue equation of H is

$$H|\Phi_\lambda\rangle = \lambda|\Phi_\lambda\rangle, \tag{13}$$

where $\lambda = \pm 1$ are the photon helicity eigenvalues and $|\Phi_\lambda\rangle$ are two eigenvectors which span a Hilbert space $\mathcal{H}$.

An $(n+1)$-dimensional Hilbert space $\mathcal{H}$ is isomorphic to an $(n+1)$-dimensional complex space C^{n+1}. The so-called projective Hilbert space $\mathcal{P}$ is consisted by rays in $\mathcal{H}$. A ray is an equivalent class of states up to overall normalization and phase, so if $|\Phi\rangle = c|\Phi'\rangle$ (where c is an arbitrary non-zero complex number), then $|\Phi\rangle = |\Phi'\rangle$. $\mathcal{P}$ is isomorphic to n-dimensional complex projective space CP^n [13]. In our case, $n = 2$, so $\mathcal{H}$ is C^2 and its projective space $\mathcal{P}$ is CP^1. $CP^1 = S^2$ is just a sphere (Poincaré sphere [9]).

Solving equation (13), we have

$$\begin{aligned} |\Psi_\pm\rangle &= e^{\mp i|\lambda|t}|\Phi_\pm\rangle \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} \pm i \\ -1 \\ 0 \end{pmatrix} \exp \pm ik(\pm x_3 - t), \end{aligned} \tag{14}$$

where signs $\pm$ stand for two circular polarized states of the photon with helicity ± 1 and $k = |\lambda| = 1$.

Any polarized states of a photon can be expressed as a superposition of two circular polarized states:

$$|\Psi\rangle = a_+|\Psi_+\rangle + a_-|\Psi_-\rangle. \tag{15}$$

In the helicity representation, the expression of a state vector at $t = 0$ is

$$|\Psi(0)\rangle = \begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}, \tag{16}$$

where θ is the angle between the directions of $|\Psi\rangle$ and $|\Psi_+\rangle$. Its evolution in time t is

$$|\Psi(t)\rangle = e^{i(\varepsilon_{ijk}\partial_j)t}\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} = \begin{pmatrix} e^{ikt}\cos(\theta/2) \\ e^{-ikt}\sin(\theta/2) \end{pmatrix}. \tag{17}$$

Equation (17) describes the evolution of photon systems in the projective space $\mathcal{P}$, *i.e.*, on the Poincaré sphere S^2. Thus, using the similar discussion of the first example in [4], we have

$$\begin{aligned} \beta &= \int_0^\tau (e^{-ikt}\cos(\theta/2), e^{ikt}\sin(\theta/2)) \\ &\quad \times i\begin{pmatrix} ike^{ikt}\cos(\theta/2) \\ -ike^{-ikt}\sin(\theta/2) \end{pmatrix} dt \\ &= -\int_0^\tau (\cos^2(\theta/2) - \sin^2(\theta/2))k\,dt. \end{aligned} \tag{18}$$

Integrating equation (18) in a period $\tau = 2\pi/k$ and up to the ambiguity of adding $2\pi n$, we obtain

$$\beta = 2\pi(1 - \cos\theta). \tag{19}$$

Equation (19) is just the AA phase of photon systems. It, indeed, is the solid angle subtended by a curve traced on the Poincaré sphere S^2.

4 The geometric phase along a spatial helix

We can even calculate geometric phases of photon systems more intuitively in ordinary geometry contexts. Image a beam of polarized light propagating along a spatial helix. For convenience, let us set two coordinates: rest one $\mathcal{O}$ and local one $\mathcal{O}'$. $\mathcal{O}'$ moves together with the photon.

In $\mathcal{O}$ the basis vectors, position vectors and partial differential operators are denoted as

$$\begin{aligned} &\mathbf{e}_i\ (i = 1, 2, 3); \\ &\mathbf{x} = x_i\mathbf{e}_i; \\ &\partial = \partial_i\,\mathbf{e}_i\ (\partial_i \equiv \partial/\partial x_i) \end{aligned} \tag{20}$$

respectively. We have adopted the summation convention here.

In $\mathcal{O}'$, these are

$$\begin{aligned} &\mathbf{e}'_i\ (i = 1, 2, 3); \\ &\mathbf{x}' = x_i\mathbf{e}_i; \\ &\partial' = \partial'_i\,\mathbf{e}'_i\ (\partial'_i \equiv \partial/\partial x'_i). \end{aligned} \tag{21}$$

The transformation rules between $\mathcal{O}$ and $\mathcal{O}'$ are

$$x_i = v_{ij}x'_j; \qquad \partial_i = v_{ji}\partial'_j; \qquad \psi_i = v_{ji}\psi'_i, \tag{22}$$

where v_{ij} are the transformation matrix elements:

$$v_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j. \tag{23}$$

Equation (5) is the Maxwell equations in $\mathcal{O}$. Using equation (22), we have

$$\varepsilon'_{lmn}\partial'_m\psi'_n = i\partial'_t\psi'_l, \tag{24}$$

where $\varepsilon'_{lmn} = \varepsilon_{ijk}v_{li}v_{mj}v_{nk}$. This can be viewed as the Maxwell equations in $\mathcal{O}'$.

Now equation (14) in $\mathcal{O}'$ should be denoted as

$$\Psi'_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm i \\ -1 \\ 0 \end{pmatrix} \exp \pm ik(\pm x'_3 - t). \tag{25}$$

Equation (24) has an exactly same form of Schrödinger equation, so we can expect that an additional geometric phase will appear when a photon propagates along a space curve. (For simplicity, we only take the circular polarized state with helicity +1 and ignore its subscript "+" in our calculation below. There is, certainly, no difference if we take the one with −1.) That is

$$\psi_k = e^{-i\gamma(t)}v_{nk}\psi'_n. \tag{26}$$

Substituting equation (26) into equation (5) and using equation (24), we have

$$\dot{\gamma}(t) v_{mi} \psi'_m = i \dot{v}_{ni} \psi'_n. \quad (27)$$

Using the orthogonality relations of v_{ij} and orthogonality and normalization relations between $\psi'^{\dagger}$ and ψ', equation (27) becomes

$$\dot{\gamma}(t) = i v_{mi} \dot{v}_{ni} \psi'^{\dagger}_{mi} \psi'_{ni}. \quad (28)$$

This is an evolution equation of geometric phase $\gamma(t)$. Integrating it, we have

$$\gamma_c = i \int_0^T dt\, v_{mi} \dot{v}_{ni} \psi'^{\dagger}_m \psi'_n. \quad (29)$$

Using equations (22, 24), equation (29) is simplified as

$$\gamma_c = \frac{1}{2} \oint \mathbf{e}'_2 \cdot d\mathbf{e}'_1 - \frac{1}{2} \oint \mathbf{e}'_1 \cdot d\mathbf{e}'_2. \quad (30)$$

In the case of a photon propagates along a helix whose axis is along $\mathbf{e}_3$, there exist following relations between basis vectors of $\mathcal{O}$ and $\mathcal{O}'$:

$$\begin{aligned} \mathbf{e}'_1 &= -\cos\phi\, \mathbf{e}_1 - \sin\phi\, \mathbf{e}_2, \\ \mathbf{e}'_2 &= \cos\theta \sin\phi\, \mathbf{e}_1 - \cos\theta\cos\phi\, \mathbf{e}_2 + \sin\theta\, \mathbf{e}_3 \end{aligned} \quad (31)$$

where θ and ϕ are polar angles: θ is the angle between the helicity direction $\mathbf{e}'_3$ and the axis of the helix $\mathbf{e}_3$, *i.e.*, the so-called pitch angle of the helix; ϕ is the orientation angle. Substituting equation (31) into equation (30) and adding $2\pi n$, we obtain the geometric phase

$$\gamma_c = 2\pi(1 - \cos\theta). \quad (32)$$

Equation (32) is the geometric phase acquired by a photon when it evolves around a helix by one cycle. If a photon evolves along the curve n times, the geometric phase becomes

$$\gamma_c^{(+)} = 2\pi n(1 - \cos\theta), \quad (33)$$

where we have added superscript (+) on, to point out that the geometric phase (Eq. (33)) is concerned in the circular polarized light with helicity +1. It is easy to show

$$\gamma_c^{(-)} = -\gamma_c^{(+)}. \quad (34)$$

5 Discussion

1. We calculate the geometric phase of the photon system by using the Maxwell equations in Schrödinger-like form and the ordinary geometry method. This approach provides a unified way to discuss the geometric phase in both of massless or massive particle systems. In addition, the calculation is clear and simple.

2. The parameter space is no long necessary in the AA formulation. The evolution of the photon along a spatial curve can be caused by an optical fiber, a series of polarizers or even a curved space. We need not to limit the discussion about photon's geometric phases only in the case of optical fiber. Thus, ambiguities related to optical fibers [12,13] can be avoided.

3. All concepts and methods used in this paper are purely classical. This will probably indicate that the geometric phase of the photon system may be a classical phenomenon. In order to see that, let us apply the result equations (33, 34) to the case a helically wound optical fiber in which the photon propagates. Consider a beam of linear polarized light to inject into such a optical fiber and denote its input state as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle). \quad (35)$$

Its output state can be denoted as

$$\begin{aligned} |\psi'\rangle &= \frac{1}{\sqrt{2}}(\exp(i\gamma^+)|+\rangle + \exp(i\gamma^-)|-\rangle) \\ &= \frac{1}{\sqrt{2}}(\exp(i\gamma^+)|+\rangle + \exp(-i\gamma^+)|-\rangle). \end{aligned} \quad (36)$$

There is a factor appearing in the output state:

$$|\langle\psi|\psi'\rangle|^2 = \frac{1}{4}[\exp(i\gamma^+) + \exp(-i\gamma^+)]^2 = \cos^2\gamma^+. \quad (37)$$

According to Malus's law in optics, equation (37) means that the polarized plan of light undergoes a rotation by an angle γ^+. So the physical meaning of the geometric phase γ is classically an angle. This is just the result in [7,8], however, our method is purely classical.

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